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Co-Symplectic Geometry and Co-Lagrangian Subspaces

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A new natural structure on the tangent spaces of a co-tangent bundle is introduced and some of its properties are investigated. This structure is based on a symmetric bilinear form and leads to a geometry that is, in many respects, analogous to the symplectic geometry. The new structure can thus justifiably be called co-symplectic geometry. The null structure of co-symplectic vector spaces is investigated in detail. It is found that the manifold of all maximally isotropic subspaces of a co-symplectic vector space is a homogeneous compact manifold of dimension $\frac{1}{2}n(n-1)$ consisting of two diffeomorphic components and having fundamental group $Z_2 \oplus Z_2$. A representation of the fundamental group of this manifold is explicitly constructed in terms of quadrupoles of co-Lagrangian subspaces.

1. INTRODUCTION

Symplectic geometry plays a central role in many recent important developments in theoretical physics. [See, for example, Guillemin and Sternberg (1977, 1984) and the extensive bibliographies therein.] The reason for this is not difficult to establish. Space-time is the arena in which physical phenomena occur. Any description of those phenomena must therefore be given in terms of space-time itself, or else in terms of some suitable geometric superstructure constructed upon it, like the configuration space, the event space, the phase space, or any of the several manifolds that are in use in the different branches of mathematical physics. Furthermore, since physics is interested in the way in which a system develops in time, a description of the system will resort normally to the use of differential equations. This

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requires a further extension of the adopted geometric structure or superstructure to some kind of bundle, which is the natural environment for differential equations. It is here that symplectic geometry enters the picture: the symplectic structure is a natural structure on these bundles that is useful for the discussion of many of the equations of mathematical physics in geometric terms. It is not surprising, therefore, to find it at the heart of so many of the theories of mathematical physics. What is perhaps surprising is that it has taken so long to recognize the central role of the symplectic geometry and to give it the individual attention and study that it deserves.

But the symplectic structure is not the only structure that is natural to the fiber bundles of physics. There is another that is equally natural and equally useful, based not on an antisymmetric bilinear form on the cotangent bundle, but on a symmetric one. This replacement of the fundamental form by one of opposite symmetry leads to a geometry that, in many respects, is directly the analog of the symplectic structure on which it is modeled. Many concepts and theorems of the new geometry can be carried over directly from the old, albeit with important modifications. Its theory can thus be developed along parallel lines. For this reason, we propose to call the new geometry *co-symplectic geometry.*

It would be a mistake to regard co-symplectic geometry as a complete newcomer to physics. Rather, like its symplectic counterpart, it occurs in a natural way in many structures of interest (see, for example, Schönberg, 1957*a*,*b*, 1958; Bohm and Hiley, 1983; Frescura and Hiley, 1984). In this sense, it can be said to be latent, or implicit, in these structures. But like its symplectic cousin, which had to wait a considerable time before its intrinsic importance was recognized, co-symplectic geometry so far does not appear to have been identified as an independent structure worthy of separate study.

The usefulness of introducing a co-symplectic structure into the cotangent bundle is that it allows the immediate geometrization of a number of familiar structures in physics which cannot be geometrized easily by the methods of symplectic geometry. Among these are the fermionic operators of quantum field theory, and the Killing vector fields associated with symmetries and their corresponding conservation laws. The co-symplectic geometry, of course, is not a replacement for or a competitor against symplectic geometry. Rather, it is complementary to it, and it is foreseen that both structures will have to be used in conjunction to achieve a full geometrization of physics.

In this paper, we present a preliminary study of the co-symplectic geometry. We confine ourselves to the algebraic foundations of the theory. The definition and investigation of co-symplectic manifolds is left to a later publication. Accordingly, we begin in Section 2 with a definition of the cosymplectic vector space and define also a canonical basis for the space, which we call a co-symplectic basis. In Section 3, we introduce the co-symplectic group and its Lie algebra. Since later developments require the use of the conformal co-symplectic group and its Lie algebra, we also discuss these briefly.

Many important results and applications of the symplectic geometry hinge on the notion of Lagrangian subspaces. Also of importance in this connection are the properties of the manifold of all Lagrangian subspaces. The many analogies that exist between symplectic and co-symplectic geometries indicate that similar developments will also be of importance in the context of the later. Accordingly, we define co-Lagrangian subspaces, and establish some elementary properties of the manifold of all co-Lagrangian subspaces in Section 4; the conditions for two co-Lagrangian subspaces to be transversal are established in Section 5, and the manifold of all co-Lagrangian subspaces transversal to a given co-Lagrangian subspace is parametrized in Section 6. Finally, in Section 7, we establish some basic topological properties of the manifold of all co-Lagrangian subspaces.

2. DEFINITION OF COSYMPLECTIC VECTOR SPACE

Let E be a finite-dimensional real vector space, and E^* its linear dual. Denote their direct sum $E^* \oplus E$ by V. For the sake of definiteness, E may be interpreted as the configuration space of classical mechanics. The space E^* would then be the momentum space, and $E^* \oplus E$ the corresponding phase space.

The natural symplectic structure on V is defined by the skew-symmetric bilinear form ω: $V \times V \to \mathfrak{R}$,

$$\omega((p, q), (p', q')) := \langle p', q \rangle - \langle p, q' \rangle \tag{1}$$

Here $\langle p, q \rangle$ denotes the value of $p \in E^*$ on $q \in E$, that is,

$$\langle p, q \rangle := p(q)$$

There is another natural structure that can be defined on V. This is given by the symmetric bilinear form σ: $V \times V \to \mathfrak{R}$, where

$$\sigma((p, q), (p', q')) = \langle p', q \rangle + \langle p, q' \rangle \tag{2}$$

Because of the obvious analogy between (1) and (2), we shall call the space (V, σ) a *co-symplectic vector space*, and σ a *co-symplectic form.*

A basis for V can be obtained as follows. Select a basis ε_{q_i} of E, together with the dual basis ε_{p_i} of E,* where $i = 1, \ldots, n$, and n is the dimension of E. Then

$$\langle \varepsilon_{p_i}, \varepsilon_{q_j} \rangle = \varepsilon_{p_i}(\varepsilon_{q_j}) = \delta_{ij}$$

Put

$$e_i := (o, \varepsilon_{q_i}), \qquad e_{n+i} := (\varepsilon_{p_i}, o)$$

The vectors e_μ, with $\mu = 1, \ldots, 2n$, will be a basis for V. We shall call a basis for V formed in this way a *natural basis* for the co-symplectic space.

In terms of a natural basis, the coefficients $\sigma_{\mu\nu}$ of the matrix representing the co-symplectic form σ are given by

$$\sigma_{\mu\nu} := \sigma(e_\mu, e_\nu)$$

or

$$\sigma_{ij} = 0 = \sigma_{n+i,n+j}$$

$$\sigma_{i,n+j} = \delta_{ij} = \sigma_{n+j,i}$$

Denoting this matrix by S, we have

$$S = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \tag{3}$$

where I is the unit $n \times n$ matrix. The value of the co-symplectic form σ on $u, v \in V$ can then be written in terms of a matrix product as

$$\sigma(u, v) = u^T S v$$

Note that

$$S = \begin{bmatrix} I & I \\ I & -I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} I & I \\ I & -I \end{bmatrix}^{-1} \tag{4}$$

The matrix S is therefore equivalent, by a similarity transformation, to the matrix

$$S' = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \tag{5}$$

This would seem to indicate that the co-symplectic form σ essentially defines on V a pseudo-Euclidean metric structure with signature zero. For $n = 1$, this would then yield the ordinary hyperbolic plane. In fact, this is not so. Even though our geometry will have a good deal in common with such a pseudo-Euclidean structure, it should nevertheless not be forgotten that V has, in addition to σ, two distinguished subspaces E and E^*, which together define a unique splitting of V into the direct sum $E^* \oplus E$. This makes the co-symplectic geometry more closely analogous to the symplectic than to the pseudo-Euclidean geometry. Furthermore, the notion of "length" in the co-symplectic case is rather weak, since the set of isotropic vectors in such a space is of maximal size among the pseudo-Euclidean spaces. A classification scheme based on the notion of length therefore would not be particularly useful. Of far greater importance is the null-structure inherent in (V, σ). It is thus preferable to develop the geometry of co-symplectic spaces in a way that parallels that of symplectic spaces.

The natural bases are not the only ones in which the matrix that represents σ has the form (3). This leads us to define more generally a *co-symplectic basis* as one in which σ has the form (3). The natural bases are thus a special kind of co-symplectic basis.

3. CO-SYMPLECTIC AND CONFORMAL CO-SYMPLECTIC GROUPS AND THEIR LIE ALGEBRAS

We define the co-symplectic group $Cs(V, \sigma)$ to be the group of linear transformations on V which preserve the form σ,

$$Cs(V, \sigma) = \{A \in GL(V): \sigma(Au, Av) = \sigma(u, v), \forall u, v \in V\}$$

Since σ is symmetric with signature zero, $Cs(V, \sigma)$ is isomorphic to the orthogonal group $O(n, n; \Re)$ of real $2n \times 2n$ matrices M satisfying

$$M^T S' M = S'$$

where S' is given by (5).

Let e_μ, with $\mu = 1, \ldots, 2n$, be a natural basis for V. The group

$$\{M \in GL(2n, \Re): M^T S M = S\}$$

where S is given by (3), will be a matrix representation of $Cs(V, \sigma)$. We shall denote this group by $Cs(2n, \Re)$.

Let $A \in Cs(2n, \Re)$. It is convenient to use block matrix notation and write

$$A = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$$

where P, Q, R, S are $n \times n$ matrices. The defining relations for A then give

$$R^T P + P^T R = 0$$

$$S^T Q + Q^T S = 0$$

$$R^T Q + P^T S = I$$

The first two conditions can be met by setting the products $R^T P$ and $S^T Q$ equal to any two arbitrary skew-symmetric matrices. In particular, we can set $Q = R = 0$. Then $P = (S^T)^{-1}$, where $S \in GL(n, \Re)$. This generates a subgroup of transformations of the form

$$A = \begin{bmatrix} (S^T)^{-1} & 0 \\ 0 & S \end{bmatrix} \tag{6}$$

which are just the transformations of V induced by a general linear transformation of E. If E is identified as the configuration space of mechanics, and E^* as the momentum space, then the transformations of form (6) are just the coordinate transformations of Lagrangian mechanics. Similarly, setting $P = S = 0$, we get $Q = (R^T)^{-1}$ with $R \in GL(n, \mathfrak{R})$. This generates a subgroup of transformations which exchange the configuration and momentum spaces.

We denote by g the Lie algebra of G, and by $o(n, n; \mathfrak{R})$ that of the matrix group $O(n, n; \mathfrak{R})$. These two Lie algebras are isomorphic. Using block matrix notation, we $X \in o(n, n; \mathfrak{R})$ as

$$X = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

The defining condition

$$X^T S + SX = 0$$

then gives

$$\beta + \beta^T = 0 = \gamma + \gamma^T$$

and

$$\delta = -\alpha^T$$

with no restriction on α. Hence

$$X = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha^T \end{bmatrix}$$

and

$$\dim Cs(V, \sigma) = \dim cs(V, \sigma) = 2n^2 - n$$

We shall also need later the conformal co-symplectic group $CCs(v, \sigma)$. This is the group of linear transformations on V which preserve the form σ up to a factor,

$$CCs(V, \sigma) = \{A \in GL(V) \colon \sigma(Au, Av) = \mu_A \sigma(u, v), \forall u, v \in V\}$$

The constant μ_A might in principle be expected to depend not only on the transformation A, but also on the particular elements $u, v \in V$. However, it can be shown to depend on A alone.

The Lie algebra of $CCs(V, \sigma)$ will be denoted by $ccs(V, \sigma)$. Its defining condition is then

$$X \in ccs(V, \sigma) \Leftrightarrow \sigma(Xu, v) + \sigma(u, Xv) = \mu_X(u, v)$$

4. CO-LAGRANGIAN SUBSPACES

A maximal isotropic subspace $L \subset (V, \sigma)$ will be called a *co-Lagrangian subspace* of V. Thus, $L \subset V$ is co-Lagrangian if and only if

$$\sigma(u, v) = 0 \qquad \forall u, v \in L$$

and, if L' is isotropic, then

$$L' \supset L \Rightarrow L' = L$$

By a standard theorem in the geometry of bilinear forms (Dieudonné, 1964, Proposition 5, p. 152),

$$\dim L = \tfrac{1}{2} \dim V = n$$

Let L be some fixed co-Lagrangian subspace. We can parametrize L by setting

$$x = \begin{bmatrix} \xi \\ \eta \end{bmatrix} \theta$$

where ξ and η are fixed $n \times n$ matrices, $\theta \in \Re^n$, and

$$\operatorname{rank} \begin{bmatrix} \xi \\ \eta \end{bmatrix} = n$$

As θ ranges over $\Re^n$, x will range over the entire space L. Of course, ξ and η cannot be chosen arbitrarily. The isotropic condition on L requires that $\sigma(x, y) = 0 \; \forall x, y \in L$. Thus, for all θ, $\theta' \in \Re^n$,

$$0 = \theta^T [\xi^T \eta^T] \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} \theta' = \theta^T (\xi^T \eta + \eta^T \xi) \theta' \tag{7}$$

where we have set $y = [\begin{smallmatrix} \xi \\ \eta \end{smallmatrix}] \theta'$. This means that L is co-Lagrangian if and only if

$$0 = \xi^T \eta + \eta^T \xi$$

i.e., $\xi^T \eta$ is skew symmetric. Note that this condition can be met trivially by setting either $\xi = 0$ or $\eta = 0$. Thus, the spaces

$$\pi: \quad x = \begin{bmatrix} 0 \\ I \end{bmatrix} \theta; \qquad \tilde{\pi}: \quad x = \begin{bmatrix} I \\ 0 \end{bmatrix} \theta \tag{8}$$

with $\theta \in \Re^n$, are co-Lagrangian.

Denote by $C\mathscr{L}(V, \sigma)$ the set of all co-Lagrangian subspaces in V. The orthogonal group $Cs(V, \sigma)$ acts transitively on $C\mathscr{L}(V, \sigma)$ (Dieudonné, 1964, p. 153). The stationary group G_π of the subspace π given by (8) consists of all transformations A of the form

$$A = \begin{bmatrix} (M^T)^{-1} & 0 \\ M^T U & S \end{bmatrix}$$

where M is any nonsingular $n \times n$ matrix and U is any skew-symmetric matrix. Thus, $C\mathscr{L}(V, \sigma)$ is the homogeneous space

$$C\mathscr{L}(V, \sigma) \approx O(V, \sigma)/G_\pi$$

and

$$\dim C\mathscr{L}(V, \sigma) = \tfrac{1}{2}n(n-1)$$

Since $C\mathscr{L}(V, \sigma)$ is a submanifold of the Grassmannian manifold of all subspaces in V of dimension n, which manifold is compact, we conclude that $C\mathscr{L}(V, \sigma)$ is a homogeneous compact manifold. In fact, it is well known (Porteous, 1969, Theorem 12.12, p. 233) that $\mathscr{C}\mathscr{L}(V, \sigma)$ is diffeomorphic to $O(n, \mathfrak{R})$, and that any co-Lagrangian subspace L has the representation

$$L: \quad x = \begin{bmatrix} I - \mathscr{O} \\ I + \mathscr{O} \end{bmatrix} \theta$$

where $\mathscr{O}$ is an $n \times n$ orthogonal matrix. This representation defines a diffeomorphism of $O(n, \mathfrak{R})$ onto $\mathscr{C}\mathscr{L}(V, \sigma)$. Thus, $\mathscr{C}\mathscr{L}(V, \sigma)$ has two diffeomorphic components, $\mathscr{C}\mathscr{L}^{\pm}(V, \sigma)$, each corresponding to a choice of orientation of $\mathfrak{R}^n$. The cohomology ring of $O(n, \mathfrak{R})$ is well known (Steenrod, 1957). The symplectic approach to this pseudo-Euclidean geometry allows new interpretations of some cohomological classes.

Note that the proper orthogonal group $Cs^+(V, \sigma)$ does not act transitively on $C\mathscr{L}(V, \sigma)$ (Dieudonné, 1964). It has two orbits, namely $\mathscr{C}\mathscr{L}^{\pm}(V, \sigma)$. We quote a result from Dieudonné (1964, p. 154), which allows us to determine when two co-Lagrangian subspaces belong to the same component of $\mathscr{C}\mathscr{L}(V, \sigma)$:

1. L, L' belong to the same component of $\mathscr{C}\mathscr{L}(V, \sigma)$ if and only if n and $r = \dim(L \cap L')$ have the same parity.
2. L, L' belong to different components of $\mathscr{C}\mathscr{L}(V, \sigma)$ if and only if n and $r = \dim(L \cap L')$ have different parities.

In particular, $r = 0$ will be taken to have even parity.

5. TRANSVERSAL CO-LAGRANGIAN SUBSPACES

Subspaces U, $W \in V$ are said to be transversal if $V = U \oplus W$. We shall now show that, given a co-Lagrangian subspace $L \in V$, another co-Lagrangian subspace L' can be found which is transversal to L. Since $Cs(V, \sigma)$ acts transitively on $C\mathscr{L}(V, \sigma)$, we can always transform L into the co-Lagrangian subspace

$$\pi: \quad x = \begin{bmatrix} 0 \\ I \end{bmatrix} \theta$$

Let L' be another co-Lagrangian subspace, and let its corresponding transform be

$$\pi'\colon \quad x=\begin{bmatrix}\xi\\ \eta\end{bmatrix}\theta$$

If L and L' are transversal, then π and π' must also be transversal. This will occur if and only if

$$\operatorname{rank}\begin{bmatrix}0 & \xi\\ I & \eta\end{bmatrix}=\dim V$$

Thus, the matrix ξ must be nonsingular, a condition that can easily be met.

Introduce now a new parametrization of L' by setting $\theta=\xi^{-1}\theta'$. Then π' becomes

$$\pi'\colon \quad x=\begin{bmatrix}I\\ \eta'\end{bmatrix}\theta'$$

where η' is skew symmetric. It is clear that any co-Lagrangian subspace π' transversal to π can be obtained in this way. The set $\mathscr{L}(\pi)$ of all co-Lagrangian subspaces transversal to π is thus parametrized by the set of all skew-symmetric $n\times n$ matrices, which topologically is $\Re^{n(n-1)/2}$. We conclude therefore as follows.

Proposition 1. The set $\mathscr{L}(L)$ of all co-Lagrangian subspaces transversal to any given L is a cell of dimension $\frac{1}{2}n(n-1)$. Furthermore, the sets $\mathscr{L}(L)$, where L ranges through the entire set $C\mathscr{L}(V,\sigma)$, define an atlas on the manifold $C\mathscr{L}(V,\sigma)$.

It is necessary for later developments also to establish whether, given two co-Lagrangian subspaces L and L' which are transversal, we can find a third L'' which is transversal to both L and L'. As before, we can assume without loss of generality that $L=\pi$. Then, since L' and L'' are both transversal to L, we can write

$$L'\colon \quad x=\begin{bmatrix}I\\ \eta'\end{bmatrix}\theta; \qquad L''\colon \quad x=\begin{bmatrix}I\\ \eta''\end{bmatrix}\theta$$

For L' and L'' also to be mutually transversal, we must have

$$\operatorname{rank}\begin{bmatrix}I & I\\ \eta' & \eta''\end{bmatrix}=\operatorname{rank}\begin{bmatrix}0 & I\\ (\eta'-\eta'') & \eta''\end{bmatrix}=2n$$

Thus,

$$\operatorname{rank}[\eta'-\eta'']=n$$

Since $\eta'-\eta''$ is skew symmetric, this condition can be met only in the case when n is even. When n is odd, no third transversal L'' can be found.

6. PROJECTIONS ONTO CO-LAGRANGIAN SUBSPACES

Let L be a fixed co-Lagrangian subspace, and let L' be a co-Lagrangian subspace transversal to it. Consider the projection P: $V \to V$ defined by

$$P(u) = 0 \,\forall u \in L, \qquad P(v) = 0 \,\forall v \in L' \tag{9}$$

We thus have an exact sequence

$$0 \to L \hookrightarrow V \xrightarrow{P} L' \to 0 \tag{10}$$

It is easily demonstrated that

$$\forall x, x' \in V, \quad \sigma(Px, x') + \sigma(x, Px') = \sigma(x, x')$$

Hence every projection P of the form (9) is an element of the Lie algebra $co(V, \sigma)$ with $\mu_P = 1$. The converse is also true: if $P \in co(V, \sigma)$, $\mu_P = 1$, and $P|_L = 0$, then there is a co-Lagrangian subspace L' which is complementary to L, such that $\mathrm{Im}(P) = L'$ and such that (10) holds. The set $\mathscr{L}(L)$ of all co-Lagrangian subspaces L' transversal to L is thus in one-to-one correspondence with the set $\{P \in co(V, \sigma)\colon \mu_P = 1 \text{ and } P|_L = 0\}$.

These observations enable us to give a coordinate-free description of the set $\mathscr{L}(V, \sigma)$. Let P be any element of $co(V, \sigma)$. Define the bilinear form Q_P on V by

$$Q_P(u, v) = \sigma(Pu, v) - \tfrac{1}{2}\mu_p \sigma(u, v) \tag{11}$$

Then Q_P can be shown to be skew symmetric. The conserve is also true: given a skew-symmetric form Q and any real constant μ, then equation (11) defines a unique element P of $co(V, \sigma)$. Note that, if $P(u) = 0$ for $u \in L$, then $Q_P(u, u) = 0$ and $Q_P(u, v) = -\frac{1}{2}\mu_P \sigma(u, v)$ for $u \in L$ and $v \in L'$. Thus, Q_P defines a pairing between L and L', giving rank $Q_P = n$. We have therefore shown the following result.

Proposition 2. The followin⸤ sets are all in one-to-one correspondence:

1. The set $\mathscr{L}(L)$ of all co-Lagrangian subspaces L' transversal to L.
2. The set $\{P \in co(V, \sigma)\colon \mu_P = 1 \text{ and } P|_P = 0\}$
3. The set of all skew-symmetric bilinear forms Q on V such that

$$Q(u, v) = \tfrac{1}{2}\sigma(u, v), \qquad u \in L, \quad v \in V$$

where $L' = \ker(P - I)$, and P and Q are related by (11).

Suppose now that Q_1 and Q_2 are two skew-symmetric bilinear forms on V satisfying

$$Q_i(u, v) = -\tfrac{1}{2}\mu_{Q_i}(u, v), \qquad u \in L, \quad v \in L' \tag{12}$$

Then the difference $Q_1 - Q_2 = H$ satisfies $H(u, x) = 0$ for $u \in L$ and $x \in V$. Thus, H defines a symmetric bilinear form on the quotient space V/L. Conversely, any skew-symmetric bilinear form on V/L can be considered as a bilinear form H on $\mathfrak{R}^{2n}$ with the property that $H(u, x) = 0$ for all $u \in L$ and $x \in V$. This yields the following proposition.

Proposition 3. Let L be a fixed co-Lagrangian subspace. Then the space $\mathscr{L}(L)$ of all co-Lagrangian subspaces transversal to L is in one-to-one correspondence with the space $Q(V/L)$ of all skew-symmetric bilinear forms on V/L.

If L' is a co-Lagrangian subspace transversal to L, then we may identify V/L and L'. Given $L'' \in \mathscr{L}(L)$, the bilinear skew-symmetric form H on L' associated with L'' is given by

$$H(v_1, v_2) = \sigma(P_{L''}v_1, v_2), \qquad v_1, v_2 \in L' \tag{13}$$

where $P_{L''}$ is the projection of V onto L'' along L, i.e., the sequence

$$0 \to L \hookrightarrow V \xrightarrow{P_{L''}} L'' \to 0 \tag{14}$$

is exact. By a previous result, the bilinear form $Q_{L'}$ associated to $L' \in \mathscr{L}(L)$ vanishes on L'. Clearly, we have

$$H = (Q_{L''} - Q)|_{L'}$$

where H is defined by (13).

We note that H is nonsingular if and only if L' and L'' are transversal. In fact, if $u \in L' \cap L''$, then

$$P_{L'}(u) = P_{L''}(u) = u$$

and thus

$$Q_{L'}(u, x) = Q_{L''}(u, x) \qquad \forall x \in V$$

Hence $H(u, x) = 0$. Conversely, if $L' \cap L'' = \{0\}$, then the projection $P: L' \to L''$ is one-to-one and, from (13), we conclude that H is nonsingular on L'.

Let L', $L'' \in \mathscr{L}(L)$, i.e., L' and L'' are both transversal to L, but not necessarily to each other. Then the projections $P_{L'}$ and $P_{L''}$ along L belong to $co(V, \sigma)$, and

$$\mu_{P_{L'}} = \mu_{P_{L''}} = 1$$

Thus,

$$P_{L'} - P_{L''} \in o(V, \sigma)$$

Furthermore, $P_{L'} - P_{L''}$ is nilpotent. For, let $u \in L$. Then $P_{L'}(u) = P_{L''}(u) = 0$. Also, if $v \in L''$, then

$$P_{L'}(v) - P_{L''}(v) = P_{L'}(v) - v \in L$$

Since $P_{L'}$ and $P_{L''}$ map L onto $\{0\}$, it follows that

$$(P_{L'} - P_{L''})^2 = 0$$

as required. From the nilpotence of $P_{L'} - P_{L''}$ we obtain also

$$\exp(P_{L'} - P_{L''}) = 1 + P_{L'} - P_{L''} \in O(V, \sigma)$$

This yields the following proposition.

Proposition 4. Let L be a co-Lagrangian subspace. If L' and L'' are two co-Lagrangian subspaces transversal to L, then

$$P_{L'} - P_{L''} \in o(V, \sigma)$$

and

$$(P_{L'} - P_{L''})^2 = 0$$

Also, the map $1 + P_{L'} - P_{L''}$ is the identity on L, it carries L'' onto L', and it belongs to $o(V, \sigma)$.

7. THE FUNDAMENTAL CLASS OF $H^1(\mathscr{CL}(V, \sigma))$

Since $\mathscr{CL}(V, \sigma)$ is diffeomorphic to $O(n, \mathscr{R})$ (Porteous, 1969), we have, for $n \geq 3$,

$$H^1(\mathscr{CL}(V, \sigma)) = \pi_1(\mathscr{CL}(V, \sigma)) = Z_2 \oplus Z_2 \tag{15}$$

Now, we know that $SO(n, \mathfrak{R})$ is double covered by the spinor group, which is one-connected, and that any one-connected manifold is orientable. But a manifold M is orientable if and only if $H^1(M, Z_2) = 0$. Hence, from (15), we conclude that $\mathscr{CL}(V, \sigma)$ is not orientable.

Let M be a nonorientable manifold and suppose that $\mathscr{A} = \{U, \phi_U\}$ is an atlas on M. Then the coordinate transformations of the principal frame bundle on M are given by

$$g_{UV}(x) = \mathscr{T}(\phi_V \circ \phi_U^{-1})(x) \tag{16}$$

where $x \in \phi_U^{-1}(U \cap V)$, U, $V \in \mathscr{A}$, and $\mathscr{T}(\cdot)$ denotes the Jacobi matrix. The functions

$$\alpha_{UV}(x) = \det[g_{UV}(x)], \qquad U, V \in \mathscr{A} \tag{17}$$

are the coordinate transformations of the associated line bundle $l(M)$. Since M is not orientable, the bundle $l(M)$ is nontrivial. The functions

$$m_{UV}(x) = \operatorname{sgn}(\alpha_{UV}), \qquad U, V \in \mathscr{A} \tag{18}$$

define a Čech 1-cocycle with coefficients in $Z_2 \approx \{1, -1\}$.

Hörmander (1971, p. 156) has given a description of $H^1(\mathscr{L}ag(V, \omega))$, where $\mathscr{L}ag(V, \omega)$ is the space of all Lagrangian subspaces in (V, ω), in terms of an invariant constructed from a quadruplet of Lagrangian subspaces. This cannot be done for the case of co-symplectic subspaces, since quadruplets of mutually transversal co-Lagrangian subspaces do not exist. Furthermore, there seems to be no invariant of the type used by Hörmander, constructed from a pair of skew-symmetric bilinear forms.

Let $\mathscr{L}(L) \subset \mathscr{CL}(V, \sigma)$ be the set of all co-Lagrangian subspaces transversal to L. From Proposition 3, Section 6, we know that $\mathscr{L}(L)$ is a cell diffeomorphic to the space of all skew-symmetric bilinear forms on the quotient space V/L. Furthermore, from Dieudonné (1964, p. 154) we also know that, if n is even, $L \in \mathscr{L}(L)$, and if n is odd, $L \notin \mathscr{L}(L)$. In the latter case, L and $\mathscr{L}(L)$ belong to different components of $\mathscr{CL}(V, \sigma)$.

If $L, L' \in \mathscr{CL}(V, \sigma)$ and $L_1 \in \mathscr{L}(L) \cap \mathscr{L}(L')$, then one can identify V/L and V/L' with the complementary space L_1. A given $L_2 \in \mathscr{L}(L) \cap (L) \cap \mathscr{L}(L')$ then defines the skew-symmetric bilinear form on L_1 by

$$S_{L_2}(u, v) = \sigma(P_{L_2}u, v), \qquad u, v \in L_1 \tag{19}$$

where P_L is the projection of V onto L_2 along L. Similarly,

$$S'_{L_2}(u, v) = \sigma(P'_{L_2}u, v), \qquad u, v \in L_1 \tag{20}$$

where P'_L is the projection of V onto L_2 along L'.

The intersection $\mathscr{L}(L) \cap \mathscr{L}(L')$ has a finite number of connected components. To illustrate this, we take

$$L:\quad x = \begin{bmatrix} I \\ 0 \end{bmatrix} \xi, \qquad L':\quad x = \begin{bmatrix} I \\ S' \end{bmatrix} \xi \tag{21}$$

Then $L_1: x = [{X \atop I}]\xi$ belongs to $\mathscr{L}(L) \cap \mathscr{L}(L')$ if and only if

$$\operatorname{rank}\begin{bmatrix} I & X \\ S' & I \end{bmatrix} = 2n$$

which is equivalent to

$$\phi(X) = \det[I - S'X] \neq 0 \tag{22}$$

Denote the algebraic set

$$M_0(L, L') = \{X : \phi(X) = 0\} \tag{23}$$

Now, we find that if $X=[x_{ij}]$, $S'=[s_{ij}]$, $i, j=1,\ldots,n$, then

$$\frac{\partial\phi}{\partial x_{ij}}=U_{ij}\times S_{ij} \qquad \text{(no summation)} \tag{24}$$

where U_{ij} is the (i,j) algebraic cofactor of $I-S'X$. Equations (24) and (23) are linearly independent. Therefore, on a dense and open subset of $M_0(L, L')$, the gradient of ϕ is nonzero. Let $X_0\in M_0$ and grad $\phi(x_0)\neq 0$. Suppose U_1, U_2 are two adjacent components of $\mathscr{L}(L)\cap\mathscr{L}(L')$, i.e., $X_0\in \bar{U}_1\cap\bar{U}_2$, where $\bar{U}$ denotes the closure of U. Consider the function

$$\phi(t)=\phi(X_0+t\operatorname{grad}\phi(X_0))$$

Then $\phi(0)=0$, and $\phi'(0)=\|\operatorname{grad}\phi(X_0)\|^2$. This shows that $\phi(t)$ changes sign when passing through $t=0$. We thus conclude the following result.

Proposition 5. If U_1, U_2 are two components of $\mathscr{L}(L)\cap\mathscr{L}(L')$ such that grad $\phi(X_0)\neq 0$ for some $X_0\in\bar{U}_1\cap\bar{U}_2$, then the function $\phi(X)$ has different signs on U_1 and U_2.

We now take any two co-Lagrangian subspaces L and L', and choose L_1, $L_2\in\mathscr{L}(L)\cap\mathscr{L}(L')$. By the homogeneity of $\mathscr{C}\mathscr{L}(V,\sigma)$, we can assume once again without loss of generality that

$$L\colon\ x=\begin{bmatrix} I\\ S\end{bmatrix}\xi, \qquad L'\colon\ x=\begin{bmatrix} I\\ S'\end{bmatrix}\xi \tag{25}$$

$$L_1\colon\ x=\begin{bmatrix} X\\ I\end{bmatrix}\xi, \qquad L_2\colon\ x=\begin{bmatrix} I\\ Y\end{bmatrix}\xi \tag{26}$$

where $\xi\in\Re^n$. Then we define

$$(L, L'; L_1, L_2)=\operatorname{sgn}\det(I-SX)(I-S'X)(I-SY)(I-S'Y) \tag{27}$$

Clearly, $(L, L'; L_1, L_2)$ is a locally constant function on L_1, $L_2\in\mathscr{L}(L)\cap\mathscr{L}(L')$, with values in $Z_2\approx\{1,-1\}$. The following obvious identities show furthermore that $(L, L'; L_1, L_2)$ defines a Čech 1-cocycle over Z_2,

$$(L, L'; L_1, L_2)=(L, L'; L_2, L_1)$$

$$(L, L'; L_1, L_2)(L, L'; L_2, L_3)(L, L'; L_3, L_1)=1$$

where L_1, L_2, $L_3\in\mathscr{L}(L)\cap\mathscr{L}(L')$. Denote the corresponding cohomology class by ω. We shall now demonstrate the following result.

Proposition 6. The cocycle ω generates the cohomology class $H^1(\mathscr{C}\mathscr{L}(V,\sigma))$.

Suppose $\omega=d\alpha$, $\alpha\in H^0(\mathscr{C}\mathscr{L}(V,\sigma))$. Then the cocycle ω either defines a function, or it is constant on $\mathscr{C}\mathscr{L}(v,\sigma)$. Suppose

$$\mathscr{L}(L)\cap\mathscr{L}(L')\cap\mathscr{L}(K)\cap\mathscr{L}(K')\neq\varnothing$$

then the corresponding zero sets $M_0(L, L')$ and $M_0(K, K')$ are distinct. Let U be a component of $\mathscr{L}(L) \cap \mathscr{L}(L')$ such that $U \cap M_0(K, K') \neq 0$. Since the regular points of $M_0(K, K')$ are dense in $M_0(K, K')$ and U is open, there is a regular point X_0 of $M_0(K, K')$ in U. According to Proposition 5, the function $(K, K'; K_1, K_2')$ changes its sign in U. Since U is a connected component of $\mathscr{L}(L) \cap \mathscr{L}(L')$, the function $(L, L'; L_1, L_2')$ is constant on U. Thus, either the cocycle ω cannot define a function, or it is not the coboundary of any cocycle, contrary to the initial hypothesis. This completes the proof.

Propositions 3.1 and 3.2 of Guillemin and Sternberg (1977) for the Lagrangian subspaces of symplectic geometry can be demonstrated without difficulty for the co-Lagrangian subspaces of co-symplectic geometry.

Proposition 7. Let R be an isotropic subspace of (V, σ). Then $W = R^{\perp}/R$ is a co-symplectic vector space, and the map

$$\rho(X) = X \cap R^{\perp}/R$$

sends $\mathscr{CL}(V, \sigma)$ into $\mathscr{CL}(W, \sigma_W)$, where σ_W is the induced co-symplectic bilinear form on W.

The proof of this proposition follows directly from the analogous symmplectic case. See Guillemin and Sternberg (1977, p. 131) and also Dieudonné (1964, p. 154). The case in which $\dim R = 1$ is of particular importance. We have

$$\dim W = \dim R^{\perp}/R = 2n - 2$$

Put $\mathscr{S}_R = \{L \in \mathscr{CL}(V, \sigma);\ L \supset R\}$. Then we have the following result.

Proposition 8. The set $\mathscr{S}_R$ is a submanifold of codimension $(n-1)$ in $\mathscr{CL}(V, \sigma)$. The map ρ, when restricted to $\mathscr{CL}(V, \sigma)\backslash\mathscr{S}_R$, is a smooth map, making $\mathscr{CL}(V, \sigma)\backslash\mathscr{S}_R$ into a fiber bundle over $\mathscr{CL}(W, \sigma_W)$ with fiber diffeomorphic to $\mathfrak{R}^n$.

Clearly $\mathscr{S}_R$ has codimension $(n-1)$ in $\mathscr{CL}(V, \sigma)$ if and only if $\mathscr{S}_R \cap \mathscr{L}(L)$, which is nonempty, has the codimension $(n-1)$ in $\mathscr{L}(L)$ for all L. By an appropriate transformation from $O(V, \sigma)$, we can transform $\mathscr{L}(L)$ into $\mathscr{L}(\pi)$, where

$$\pi: \begin{bmatrix} 0 \\ I \end{bmatrix} \xi$$

and

$$\mathscr{L}(\pi) = \left\{ L \in \mathscr{CL}(V, \sigma); \qquad L: \quad x = \begin{bmatrix} I \\ S \end{bmatrix} \xi \right\}$$

Suppose that under this transformation R becomes

$$x = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} t, \qquad t \in \mathfrak{R}$$

Then R is isotropic if and only if $R_1, R_2 \in \mathfrak{R}^n$ are perpendicular with respect to the standard Euclidean product in $\mathfrak{R}^n$. Clearly $L \in \mathscr{S}_R$ if and only if for some $\xi \in \mathfrak{R}^n$,

$$\begin{bmatrix} I \\ S \end{bmatrix} \xi = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$$

Thus,

$$SR_1 = R_2 \tag{28}$$

Since R_1 is perpendicular to R_2 and S is a skew-symmetric $n \times n$ matrix, equation (28) determines $(n-1)$ linearly independent equations in the entries of $\mathscr{S}$. Hence codim $\mathscr{S}_R = n-1$.

We now show the map ρ: $\mathscr{S}_R \to \mathscr{C}\mathscr{L}(V, \sigma)$ given by

$$L \to (R^\perp \cap L)/(R \cap L) = L/R$$

is a bijection. Recall that $R \subset L$ and $L \subset R^\perp$. Then, if $L' \subset R^\perp/R$ is co-Lagrangian, the preimage of L' by the quotient map

$$\pi_W: \quad R^\perp \to R^\perp/R$$

is an unique co-Lagrangian subspace $L \supset R$ in $R^\perp$ such that $\rho(L) = L'$.

Let us now investigate the complementary set $\mathscr{C}\mathscr{L}(V, \sigma)/\mathscr{S}_R$. For $L \not\subset R^\perp$, that is, $R \not\subset L$, the map $L \to L \cap R^\perp$ is smooth, since $\dim(L \cap R^\perp)$ is constant. Furthermore, $L \cap R^\perp$ does not contain R. Therefore the map

$$L \cap R^\perp \to (L \cap R^\perp)/(L \cap R) = (L \cap R^\perp)/\{0\}$$

is also smooth. Thus, ρ is smooth on $\mathscr{C}\mathscr{L}(V, \sigma)\backslash\mathscr{S}_R$. Again, since $L \not\supset R$, and therefore $L \not\subset R^\perp$, the intersection $L \cap R^\perp$ is an $(n-1)$-dimensional isotropic subspace in $R^\perp$ which does not contain R. This implies that the image $(L \cap R^\perp)/R$ is an $(n-1)$-dimensional isotropic or co-Lagrangian subspace in W. We conclude that the preimage $\pi_W^{-1}(K)$ of any co-Lagrangian subspace K in W is an $(n-1)$-dimensional isotropic subspace in $R^\perp$.

Given an $(n-1)$-dimensional isotropic subspace $L' \subset R^\perp$, the set of all co-Lagrangian subspaces L in V such that $L \cap R^\perp = L'$ can be parametrized as follows. Note that

$$(L')^\perp = L^\perp + R = L + R$$

Thus, $\dim(L')^\perp = n+1$. Since $L' \subset (L')^\perp$, take a complementary subspace to L' in $(L')^\perp$, say $Z \subset (L')^\perp$. Then

$$(L')^\perp = L' \oplus Z$$

and we have $\dim Z = 2$. Because $(L')^{\perp} = L \oplus R$, every co-Lagrangian subspace in $(L')^{\perp}$ is determined uniquely by a one-dimensional subspace in Z, with the exception of the subspace R. The set of all one-dimensional subspaces in Z is a circle. A circle with one point removed is an open segment. Thus, the preimage $\rho^{-1}(L')$, with $L' \in \mathscr{CL}(W)$, is diffeomorphic to $\mathfrak{R}^{n-1} \times$ the open segment, which gives $\mathfrak{R}^n$. It follows then that the fibration $\mathscr{CL}(V, \sigma)/\mathscr{S}_R$ over $\mathscr{CL}(W)$ is locally trivial.

8. CONCLUSION

In this paper, we have defined the co-symplectic geometry and have investigated the null structure of a co-symplectic vector space (V, σ). The set $\mathscr{CL}(V, \sigma)$ of all maximal isotropic subspaces of V, which we call co-Lagrangian subspaces, is a homogeneous compact manifold of dimension $\frac{1}{2}n(n-1)$ consisting of two diffeomorphic components. The set $\mathscr{L}(L)$ of all co-Lagrangian subspaces transversal to a given co-Lagrangian subspace L is a cell of dimension $\frac{1}{2}n(n-1)$ and can be parametrized by the skew symmetric bilinear forms on V/L. The sets $\mathscr{L}(L)$, with L ranging through the entire set $\mathscr{CL}(V, \sigma)$, define an atlas on $\mathscr{CL}(V, \sigma)$. The fundamental group of $\mathscr{CL}(V, \sigma)$ is $Z_2 \oplus Z_2$.

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